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The Curious Case of Pascal's Triangle

Pascal's Triangle and the Binomial Theorem

LEARNING GOALS

In this lesson, you will:

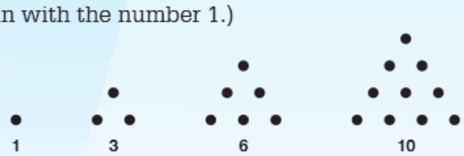
- Identify patterns in Pascal's Triangle.
- Use Pascal's Triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.
- Extend the Binomial Theorem to expand binomials of the form $(ax + by)^n$.

KEY TERM

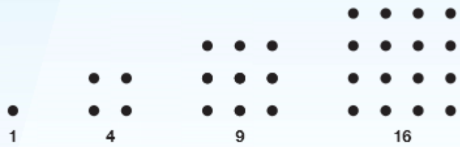
- Binomial Theorem

Some sets of numbers are given special names because of the interesting patterns they create. A polygonal number is a number that can be represented as a set of dots that make up a regular polygon. For example, the number 3 is considered to be a polygonal number because it can be represented as a set of dots that make up an equilateral triangle, as shown.

More specifically, the polygonal numbers that form equilateral triangles are called the triangular numbers. The first four triangular numbers are shown. (Note that polygonal numbers always begin with the number 1.)



The square numbers are polygonal numbers that form squares. The first four square numbers are shown.

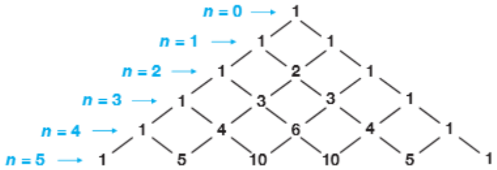


Can you determine the first four pentagonal numbers? How about the first four hexagonal numbers?

PROBLEM 1 So Many Patterns, So Little Time

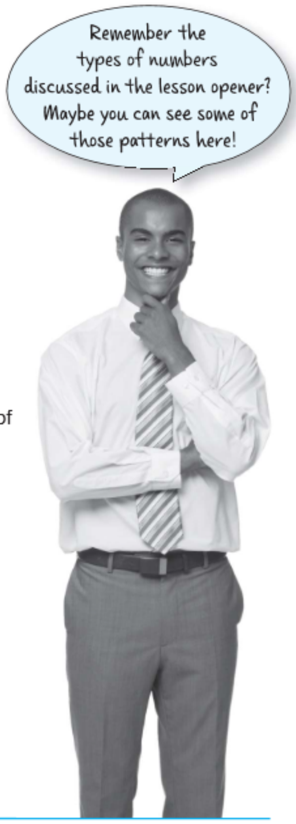
There is an interesting pattern of numbers that makes up what is referred to as Pascal's Triangle.

The first six rows of Pascal's Triangle are shown, where $n = 0$ represents the first row, $n = 1$ represents the second row, and so on.



1. Analyze the patterns in Pascal's Triangle.

   a. Describe all the patterns you see in Pascal's Triangle.



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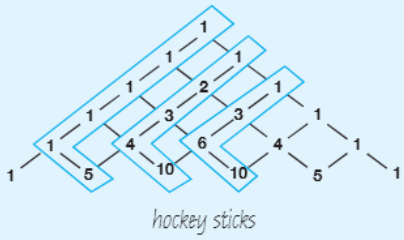
   b. Complete the rows for $n = 6$ and $n = 7$ in the diagram of Pascal's Triangle. Describe the pattern you used.

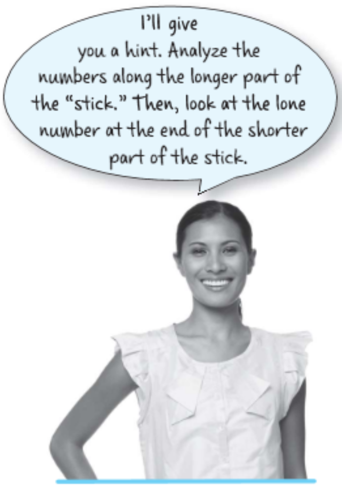
2. Brianna loves hockey. In fact, Brianna is so obsessed with hockey that she drew "hockey sticks" around the numbers in Pascal's Triangle. Lo and behold, she found a pattern! Her work is shown.

Brianna



a. Describe the pattern shown by the numbers inside the hockey sticks that Brianna drew.

b. Sketch two more hockey sticks that include numbers that have the same pattern described in part (a).



3. Drew and Latasha analyzed Pascal's Triangle, and each described a pattern.

Drew

The sum of the numbers in each row is equal to 2^n, where $n = 0$ represents the first row.

Latasha

If I alternate the signs of the numbers in any row after the first row and then add them together, their sum is 0.

Who's correct? Either verify or disprove each student's work.

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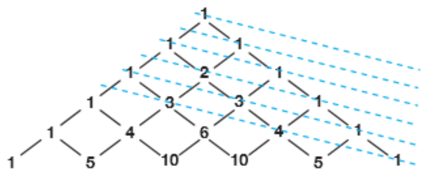
4. Consider the numbers along the dashed lines shown.



a. Write the sequence for the sum of numbers along each dashed line.

b. Explain how the sums of numbers along the dashed lines in Pascal's Triangle can be linked to a well-known sequence of numbers.

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The patterns shown in Pascal's Triangle have many uses. For instance, you may have used Pascal's Triangle to calculate probabilities. Let's explore another use for Pascal's Triangle when raising a binomial to a positive integer.

5. Multiply to expand each binomial. Write your final answer so that the powers of *a* are in descending order.

a. $(a + b)^0 =$

b. $(a + b)^1 =$

c. $(a + b)^2 =$

d. $(a + b)^3 =$

e. $(a + b)^4 =$

6. Analyze your answers to Question 5.

   a. Compare the coefficients of each answer with the numbers shown in Pascal's Triangle. What do you notice?

   b. What do you notice about the exponents of the *a*- and *b*-variables in each expansion?

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   c. What do you notice about the sum of the exponents of the *a*- and *b*-variables in each expansion?

7. Use Pascal's Triangle to expand each binomial.

   a. $(a + b)^5 =$

   b. $(a + b)^6 =$

   c. $(a + b)^7 =$

PROBLEM 2 Binomial Theorem Delirium!

What if you want to expand a binomial such as $(a + b)^{15}$? You could take the time to draw that many rows of Pascal's Triangle, but there is a more efficient way.

Recall that the factorial of a whole number n, represented as $n!$, is the product of all numbers from 1 to n.

1. Perform each calculation and simplify.

   a. $5! =$

   b. $2!3! =$

   c. $\frac{5!}{3!} =$

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You may have seen the notation $\binom{n}{k}$ or ${}_nC_k$ when calculating probabilities in another course. Both notations represent the formula for a *combination*. Recall that a combination is a selection of objects from a collection in which order does not matter. The formula for a combination of k objects from a set of n objects for $n \geq k$ is shown.

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$

Calculate $\binom{4}{2}$, or ${}_4C_2$.

$\binom{n}{k} = {}_nC_k = \dfrac{n!}{k!(n-k)!}$	Write the formula for a combination.
$n = 4$ and $k = 2$	Identify n and k.
$\binom{4}{2} = \dfrac{4!}{2!(4-2)!}$	Substitute the values for n and k into the formula.
$\binom{4}{2} = \dfrac{4!}{2!(4-2)!} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)}$	Write each factorial as a product.
$\binom{4}{2} = \dfrac{4!}{2!(4-2)!} = \dfrac{4 \cdot 3 \cdot \cancel{2 \cdot 1}}{(2 \cdot 1)\cancel{(2 \cdot 1)}}$	Divide out common factors.
$\binom{4}{2} = \dfrac{4!}{2!(4-2)!} = \dfrac{4 \cdot 3 \cdot \cancel{2 \cdot 1}}{(2 \cdot 1)\cancel{(2 \cdot 1)}} = \dfrac{12}{2} = 6$	Simplify.

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2. Explain why n must be greater than or equal to k in the formula for a combination.

3. Perform each calculation and simplify.

a. $\binom{5}{1} =$

b. ${}_7C_4 =$

4. Sarah and Montel's teacher asks each student to use Pascal's Triangle to calculate ${}_6C_3$. Their answers and explanations are shown.

Sarah

I can calculate ${}_nC_k$ by looking at the kth number (from left to right) in the nth row of Pascal's Triangle. So, ${}_6C_3$ is equal to 10.

Montel

I can calculate ${}_nC_k$ by looking at the $(k + 1)$th number (from left to right) in the $(n + 1)$th row of Pascal's Triangle. So, ${}_6C_3$ is equal to 20.

Who is correct? Explain your reasoning.

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The **Binomial Theorem** states that it is possible to extend any power of $(a + b)$ into a sum of the form shown.

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$$

5. Use the Binomial Theorem to expand $(a + b)^{15}$. You can use your calculator to determine the coefficients.

$(a + b)^{15} =$

Suppose you have a binomial with coefficients other than one, such as $(2x + 3y)^5$. You can use substitution along with the Binomial Theorem to expand the binomial.

You can use the Binomial Theorem to expand $(a + b)^5$, as shown.

$$(a + b)^5 = \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$$
$$= a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$$

Now consider $(2x + 3y)^5$.

Let $2x = a$ and let $3y = b$.

You can substitute $2x$ for a and $3y$ for b into the expansion for $(a + b)^5$.

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4(3y)^1 + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)^1(3y)^4 + (3y)^5$$
$$= 32x^5 + 5(16x^4)(3y) + 10(8x^3)(9y^2) + 10(4x^2)(27y^3) + 5(2x)(81y^4) + 243y^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 8105xy^4 + 243y^5$$

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6. Use the Binomial Theorem and substitution to expand each binomial.

a. $(3x + y)^4$

b. $(x - 2y)^6$

Be prepared to share your solutions and methods.